is defined as part of the boundary, the unboundedness problem will be avoided for interior points near the singularity

The stability problem may be discussed along the lines presented by Forsythe and Wasow [2] The term $1 - (\alpha k/2r)$ will be positive if $\alpha < (2r/k)$ If this latter condition is met and if α,β,k,h,r are all positive and finite, Eq (7) merely represents $\psi(z,r)$ as a weighted average of four surrounding points Since the boundary is specified and finite, and since all derivatives are bounded in the region under consideration, all interior points must be finite: $0 \leq m \leq \psi(z,r) \leq M \leq \infty$ Thus, the difference equation should be stable throughout the region interior to the boundary

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Three-Dimensional Boundary Layers with a Normal Wall Velocity

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1 Introduction

COOKE[1] applied the Stewartson[2] transformation, when the crossflow is small, to the three dimensional boundary-layer equations in compressible flow and thereby reduced them to the equations describing a certain three-dimensional incompressible flow In the present note we achieve a similar result for the case where there is a normal velocity at the wall, thus extending the work of the writer[3] to three-dimensional layers with small crossflow

The boundary-layer equations when the crossflow is small have been derived by Cooke,[4] using streamline coordinates, in the following form:

$$\rho\left(u\frac{\partial u}{\partial s} + w\frac{\partial u}{\partial \zeta}\right) = \rho_e u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial u}{\partial \zeta}\right) \qquad (1)$$

$$\rho\left(u\frac{\partial v}{\partial s} + w\frac{\partial v}{\partial \zeta} + \frac{uv}{r}\frac{\partial r}{\partial s}\right) = K(\rho_e u_e^2 - \rho u^2) + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial v}{\partial \zeta}\right) \qquad (2)$$

$$\rho C_p\left(u\frac{\partial T}{\partial s} + w\frac{\partial T}{\partial \zeta}\right) = -\rho_e u u_e \frac{\partial u_e}{\partial s} + \frac{\partial}{\partial \zeta}\left(k\frac{\partial T}{\partial \zeta}\right) + \mu\left(\frac{\partial u}{\partial \zeta}\right)^2 \qquad (3)$$

$$\frac{\partial}{\partial s}(\rho r u) + \frac{\partial}{\partial \zeta}(\rho r w) = 0 \qquad (4)$$

The coordinate system (ξ, η, ζ) with corresponding velocity components (u, v, w) is such that ζ is measured normal to the surface (occupying $\zeta = 0$); $\eta = \text{const}, \zeta = 0$ are the projections of the external streamlines on to this surface, and $\xi = \text{const}, \zeta = 0$ are their orthogonal trajectories The length element dl is then given by

$$dl^2 = h_1^2(\xi,\eta)d\xi^2 + r^2(\xi,\eta)d\eta^2 + d\zeta^2$$

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Cooke defined ds as the length element along the curves $\eta = \text{const}, \zeta = 0$ Thus, $ds = h_1 d\xi$ and the s and ξ directions are the same In Eqs (1–4), $K = -(\partial h_1/\partial \eta)/h_1 r$ and ρ, T, μ, C_p, and k are the density, temperature, viscosity, specific heat, and thermal conductivity of the fluid The suffix e denotes values just outside the boundary layer, and we shall use suffixes 0 and w for values at a standard isentropic reference position and the wall, respectively

As usual, for correlation we need to make three restrictive assumptions: 1) the Prandtl number σ is unity, so that $k = C_p\mu$; 2) the viscosity is proportional to the temperature although we may allow the proportionality factor to vary with s and η; thus, we choose $\mu = (\mu_w/T_w)T$, where μ_w is related accurately to T_w; and 3) the surface is heat-insulating, $(\partial T/\partial \zeta)_{\zeta=0} = 0$ The boundary conditions for Eqs (1–4) are

$$u = v = 0 \qquad w = w_w(s,\eta) \qquad \partial T/\partial \zeta = 0 \qquad \text{at } \zeta = 0$$

$$u = u(s\,\eta) \qquad v = 0 \qquad T = T(s,\eta) \qquad \text{at } \zeta = \infty$$

Under assumptions 1) and 3), the temperature can be written in the form

$$\frac{T}{T_e} = 1 + \frac{(\gamma - 1)}{2a^2}(u_e^2 - u^2) \qquad (5)$$

where γ is the ratio of specific heats, and a is the sound speed Equation (5) is the exact solution of Eq (3), consistent with (1) and (4), but is an approximate expression for T since it is assumed that $v \ll u$

2 Analysis

The analysis that follows is similar to the one used by Gribben bearing in mind that now each unknown depends on three independent variables That in turn was based on Illingworth's[5] original form of the correlation transformation which treated the boundary layer equations written in the Von Mises' coordinates where the stream function is used as an independent variable instead of distance normal to the surface Thus, here, the first step is to transform to new independent variables (x, y, ψ) where

$$x = s \qquad y = \eta \qquad \psi = \psi(s,\eta,\zeta)$$

and ψ is defined to satisfy (4) identically, i e ,

$$\frac{\partial \psi}{\partial \zeta} = \frac{r\rho u}{\rho_0} \qquad \frac{\partial \psi}{\partial s} = -\frac{r\rho w}{\rho_0} \qquad (6)$$

In addition, we introduce the nondimensional velocity $\bar{u} = u/u_e$ when Eqs (1) and (2) become

$$\frac{\partial \bar{u}}{\partial x} = \frac{1}{u_e\bar{u}}\frac{\partial u_e}{\partial x}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + F(x,y)\frac{\partial}{\partial \psi}\left(\bar{u}\frac{\partial \bar{u}}{\partial \psi}\right) \qquad (7)$$

$$\frac{1}{r}\frac{\partial}{\partial x}(vr) = \frac{Ku_e}{\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + F(x,y)\frac{\partial}{\partial \psi}\left(\bar{u}\frac{\partial v}{\partial \psi}\right) \qquad (8)$$

In these equations the product $\mu\rho$ has been replaced by $\mu_w\rho_w$ in accordance with assumption 2), the perfect gas law, and the fact that the pressure is independent of the ψ coordinate It follows that

$$F(x,y) = \mu_w\rho_w r^2 u_e/\rho_0^2$$

The function ψ is closely related to the stream function of axisymmetric flow but is only determined from (6) to within an arbitrary function of η, say $g(\eta)$ Thus, the value of ψ at the surface is given as

$$(\psi)_{\zeta=0} = f(x,y) = -\int_0^x \frac{r\rho_w w_w}{\rho_0}dx + g(y)$$

The boundary conditions are then

$$\bar{u}[x,y,f(x,y)] = v[x,y,f(x,y)] = 0$$

$$\bar{u}(x,y,\infty) = 1 \qquad v(x,y,\infty) = 0$$

These become independent of $f(x, y)$, and the equations are simplified on introducing the transformation

$$d\theta = F(x,y)dx \qquad \phi = \psi - f(x,y)$$

Equations (7) and (8) become

$$\frac{\partial \bar{u}}{\partial \theta} + G(\theta,y)\frac{\partial \bar{u}}{\partial \phi} = \frac{1}{u_e\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right)\frac{\partial u_e}{\partial \theta} + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial \bar{u}}{\partial \phi}\right) \quad (9)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(vr) = \frac{Ku_e}{F\bar{u}}\left(\frac{\rho_e}{\rho} - \bar{u}^2\right) + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial v}{\partial \phi}\right) \quad (10)$$

in which $G(\theta, y) = r\rho_w w_w/\rho_0 F$, and the boundary conditions are $\bar{u}(\theta, y, 0) = v(\theta, y, 0) = v(\theta, y, \infty) = 0$, $\bar{u}(\theta, y, \infty) = 1$

As usual, on applying the solution (5) to the boundary layer in a perfect gas we have

$$(\rho_e/\rho) - \bar{u}^2 = [1 + (\gamma - 1)M_e^2/2](1 - \bar{u}^2) = c(a_0/a)^2(1 - \bar{u}^2)$$

where $c = 1 + (\gamma - 1)M_0^2/2$, M is the Mach number, and Eqs (9) and (10) become, finally,

$$\frac{\partial \bar{u}}{\partial \theta} + G(\theta,y)\frac{\partial \bar{u}}{\partial \phi} = c\left(\frac{a_0}{a_e}\right)^2\frac{(1 - \bar{u}^2)}{\bar{u}}\frac{\partial}{\partial \theta}(\log u) + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial \bar{u}}{\partial \phi}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(vr) = \frac{Ku_e c}{F}\left(\frac{a_0}{a}\right)^2\frac{(1 - \bar{u}^2)}{\bar{u}} + \frac{\partial}{\partial \phi}\left(\bar{u}\frac{\partial v}{\partial \phi}\right)$$

For the special case of incompressible flow denoted by suffix i with density ρ_0 and viscosity μ_0, these equations become

$$\frac{\partial \bar{u}_i}{\partial \theta} + G_i(\theta,y)\frac{\partial \bar{u}_i}{\partial \phi} = \frac{(1 - \bar{u}_i^2)}{\bar{u}_i}\frac{\partial}{\partial \theta}(\log u_i) + \frac{\partial}{\partial \phi}\left(\bar{u}_i\frac{\partial \bar{u}_i}{\partial \phi}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}(v_i r) = \frac{K_i u_{ei}(1 - \bar{u}_i^2)}{F_i\bar{u}_i} + \frac{\partial}{\partial \phi}\left(\bar{u}_i\frac{\partial v_i}{\partial \phi}\right)$$

with the same boundary conditions on $\bar{u}_i$, v_i as on $\bar{u}$, v

Hence, the two sets of equations for $\bar{u}$, v and $\bar{u}_i$, v_i have identical solutions as functions of (θ, y, ϕ), provided that

$$c\left(\frac{a_0}{a}\right)^2\frac{\partial}{\partial \theta}(\log u) = \frac{\partial}{\partial \theta}(\log u_i) \quad (11)$$

$$G(\theta,y) = G_i(\theta,y) \quad (12)$$

$$\frac{Ku_e c}{F}\left(\frac{a_0}{a_e}\right)^2 = \frac{K_i u_{ei}}{F_i} \quad (13)$$

The correlation is therefore established if the main streams of the compressible and incompressible flows are related by (11), a suitable integral of which is

$$u_i = a_0 u_e/a_e$$

the wall velocities are related by (12), yielding

$$w_{wi} = \mu_0 a_0 w_w/\mu_w a_e$$

and the values of K in the two flows are related by (13), giving

$$K_i = c\mu_0\rho_0 a_0^2 K/\mu_w\rho_w a_e^2$$

which is equivalent to Cooke's result

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Analogy between Three-Dimensionally Heated Plates and Generalized Plane Stress

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CONSIDER a thin plate of arbitrary contour C which is subject to in-plane body forces per unit of area $[X(x, y), Y(x, y)]$ where x and y are midplane coordinates Assuming that the displacements are prevented on the boundary, the principle of stationary potential energy for this system requires that[1]

$$\delta V^* = \delta\iint[V_0 - (Xu + Yv)]dxdy = 0 \quad (1)$$

where V^* is the potential energy, and V_0 is the strain energy per unit of area given by

$$V_0 = \frac{Eh}{2(1 - \nu^2)}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + 2\nu\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1 - \nu}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right] \quad (2)$$

The Euler equations corresponding to (1) and (2) are the usual two-dimensional displacement equilibrium equations which in the present case are subject to the boundary conditions

$$u = v = 0 \quad \text{on } C \quad (3)$$

If X and Y are replaced by $[-E\alpha h(\partial T/\partial x)/1 - \nu]$ and $[-E\alpha h(\partial T/\partial y)/1 - \nu]$, respectively, then the problem is identical with that of plane stress due to temperature $T(x, y)$ and no body forces [2] When the thermal gradient in the thickness direction is not severe (in which case the Bernoulli-Euler hypothesis may be employed), the above variational problem is also analogous to the more general case for which $T = T(x, y, z)$ is an even function of z The quantities X and Y are then replaced by $(\partial N_T/\partial x)/1 - \nu$ and $-(\partial N_T/\partial y)/1 - \nu$, respectively, where $N_T = \int_h E\alpha T dz$

Solutions by the Rayleigh-Ritz method require that u and v be chosen in the form

$$u = \Sigma a_n f_n(x, y)$$
$$v = \Sigma b_n g_n(x, y) \quad (4)$$

where $f_n = g_n = 0$ on C In the case of a rectangular plate, for example, the displacements u and v may be expressed in the form of a complete double Fourier series:

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$
$$v = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (5)$$

where a and b are the planform dimensions of the plate Sub-

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